

Practical Exercise to Numerical Optimization

Exercise 1:

We consider the following unrestricted optimization problem:

$$(\text{OP}) \quad f(x, y) := \frac{1}{10}x^2 + y^2 \rightarrow \min!, \quad x, y \in \mathbb{R}.$$

Compute the first derivative $\nabla f(x, y)$ and the Hessian $\nabla^2 f(x, y)$ of f . Implement them into `python` code and compute $\nabla f(x_0, y_0)$ and $\nabla^2 f(x_0, y_0)$ for $(x_0, y_0) = (0.1, 1)$.

Exercise 2:

Implement a simple gradient search to solve (OP) with constant step length $t = 0.1$. As stopping criterion we choose $\|\nabla f(x, y)\| \leq \epsilon$. How many iterations do you need for accuracy $\epsilon = 10^{-5}$ and with starting point (x_0, y_0) from Exercise 1. Make a 2D plot of the iterates (x_k, y_k) for illustration.

Exercise 3:

Implement Newton's method to solve (OP). How many iterations do you need for the same accuracy and the same initial point as in Exercise 2? Do you have an explanation for the observation?

Exercise 4:

Implement the exact line search $\alpha := \arg \min\{\varphi(\alpha) : \alpha > 0\}$, $\varphi(\alpha) := f(x + \alpha d_1, y + \alpha d_2)$ with Armijo step size control:

$$\text{ensure that } \varphi(\alpha) \leq \varphi(0) + \sigma \alpha \varphi'(0).$$

We take the initial value $\alpha = 1$, reduction factor $\beta = 0.9$ ($\alpha \rightarrow \beta \alpha$) and $\sigma = 10^{-2}$. How many iterations do you need now? Make a 2D plot of the iterates (x_k, y_k) for illustration.

Exercise 5:

Extend your algorithm from Exercise 4 to the penalty method (with penalty parameter $\alpha > 0$) in order to consider equality constraints. Adapt the optimization problem from Exercise 1 to a restricted optimization problem:

$$(\text{ROP}) \quad f(x, y) \rightarrow \min!, \quad x, y \in X := \{(x, y) \in \mathbb{R}^2 : x + y = 1.1\}.$$

Apply your algorithm to get solutions in dependence of $\alpha = 2^k$, $k = 0, \dots, 4$. You may use the solution for $\alpha = 2^k$ as initial guess for the iteration for $\alpha = 2^{k+1}$.

Exercise 6:

How many iterations needs your Newton's method for the setting in Exercise 5 for $\alpha = 2^4$?