Finite Elements

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Agenda

1. Finite Elements
   - Finite element meshes
   - Linear finite elements
   - From finite elements to linear systems
   - Numerical analysis
   - Finite elements computational remarks

Steps for a finite element discretization

Step 1: We discretize the domain Ω by a mesh Ω_h

Step 2: On Ω_h we discretize the function space $\mathcal{V} = H_0^1(\Omega)$ by a finite element space V_h

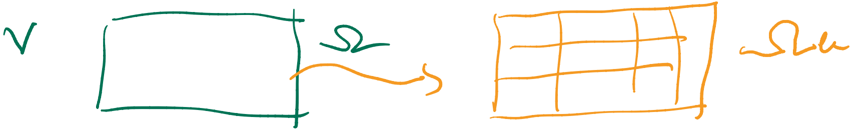
Step 3: We restrict the variational formulation to V_h

Poisson Problem

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \quad forall \phi_h in V_h$$

Step 4: We solve a linear system of equations

V_h

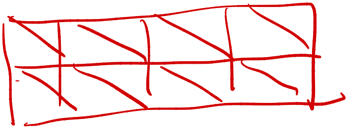


- We discretize the domain Ω by splitting it into simple **open elements**, e.g. triangles, quadrilaterals (in 2d) or tetrahedras, prisms, hexahedras, pyramids (in 3d)

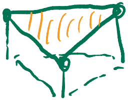
- The **finite element mesh** Ω_h is the set of all **elements**

$$\Omega_h = \{T_1, T_2, \dots, T_N\}$$

triangulation in 2D



Prism 3D

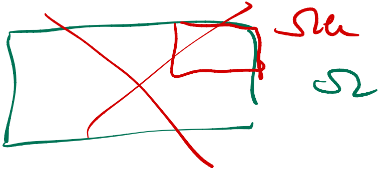


3 assumptions ⇝ need „interpolation estimates"

I: structural assumptions

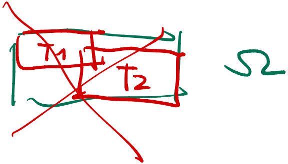
1. The union of all elements covers the domain

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{T}_i$$

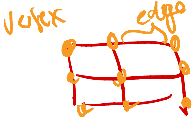


2. Two different elements never overlap

$$T_i \cap T_j = \emptyset \quad \forall i \neq j$$



3. The closure of two elements can only overlap in a **corner vertex**, an **edge** or a **face**



$$\bar{T}_i \cap \bar{T}_j = \begin{cases} x & \text{a vertex} \\ e & \text{an edge} \\ f & \text{a face} \end{cases} \quad \forall i \neq j$$

Basic rule: *triangles should look like triangles, tetrahedras should look like tetrahedras, ...*

II:Shape regularity for triangular meshes: 

1. We call a mesh shape regular, if it holds for all $T \in \Omega_h$

$$\frac{\operatorname{diam}(T)}{\rho_T} < c,$$

where ρ_T is the diameter of the largest circle in T and $\operatorname{diam}(T)$ the longest edge of T

2. Equivalent definition: All angles α in T are bound away from zero

$$\alpha \geq \alpha_0 > 0$$

with a constant $\alpha_0 > 0$.

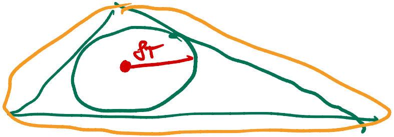
Basic rule: *triangles should have the same size more or less, ...*

III: Size regularity for triangular meshes:

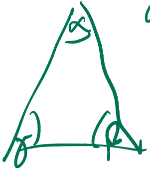
$$\max_{T \in T_h} h_T \leq c \min_{T \in T_h} h_T$$

$$\frac{\max_{T\in T_h} h_T}{\min_{T\in T_h} h_T} < C$$

1) shape regularity



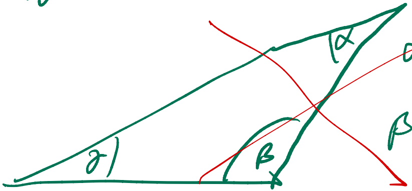
2)



$\alpha \to 0$

$\gamma, \beta > 0$

admissable



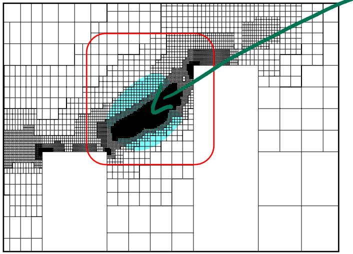
$\alpha, \gamma \to 0$

$\beta \to 180°$

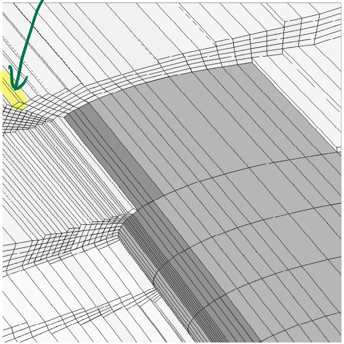
Finite Element Meshes

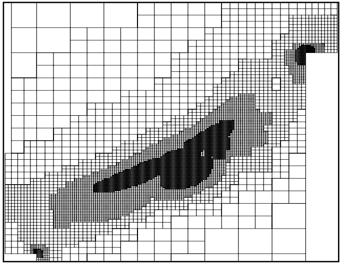
Examples

size regularity breaking down



shape regularity



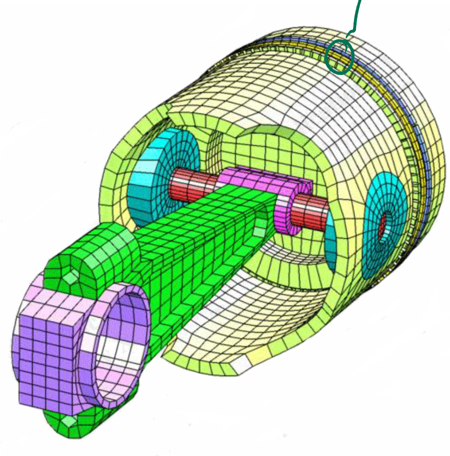


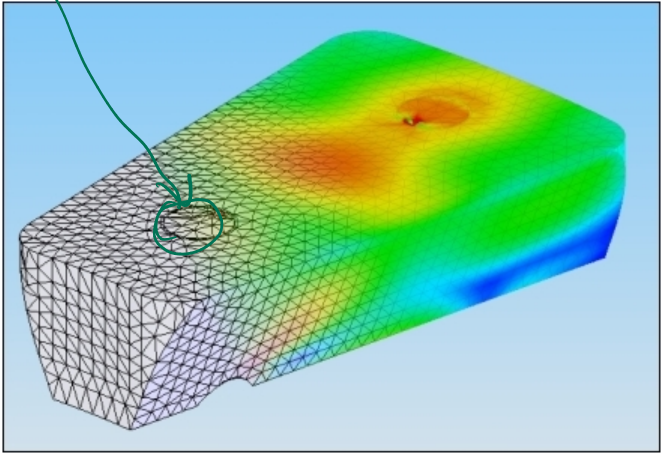
adaptive meshes

~~shape regularity~~

size ~~regularity~~

Finite Element Meshes — Examples

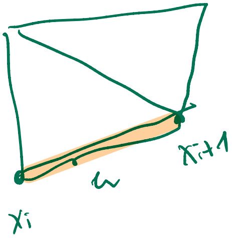




Step 2 - Discretize $\mathcal{V}$ by V_h

Local Finite Element space

- On every element $T \in \Omega_h$ define the basis functions of a simple polynomial space
- **linear finite elements** on triangles
- Triangle with the points $x^{(1)} = (0,0)$, $x^{(2)} = (h,0)$ and $x^{(3)} = (0,h)$

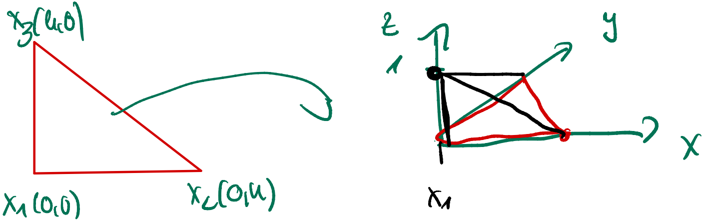


Basis function

lineare F. E. on triangles

$$\phi^{(1)}(x,y) = 1 - \frac{x}{h} - \frac{y}{h}, \quad \phi^{(2)}(x,y) = \frac{x}{h}, \quad \phi^{(3)}(x,y) = \frac{y}{h}$$

λ_1

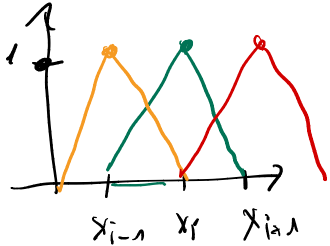


$\phi^{(1)}(0,0) = 1$

$\phi^{(1)}(h,0) = 0$

$\phi^{(1)}(0,h) = 0$

1D



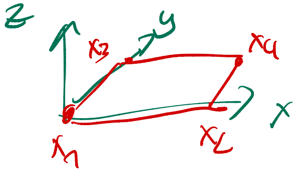
- We have basis functions on every triangle $T \in \Omega_h$
- We combine them to a global function space

$$V_h := \{\phi_h \in C(\bar{\Omega}) \mid \phi\Big|_T \in P^1 := \operatorname{span}\left(\phi_h^{(1)}, \phi_h^{(2)}, \phi_h^{(3)}\right)\}$$

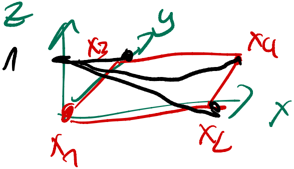
- This is called the **Lagrange basis** or **nodal basis**. It holds

$$\phi_h^{(i)} \in V_h : \quad \phi_h^{(i)}\Big|_T \in P^1, \quad \phi_h^{(i)}(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Assume that the mesh elements $T \in \Omega_h$ are quadrilaterals
- **bi-linear finite elements:**
- Let T be a quadrilateral with the points $x^{(1)} = (0,0)$, $x^{(2)} = (h,0)$, $x^{(3)} = (0,h)$, $x^{(4)} = (h,h)$.



$$\phi^{(1)}(x,y) = \left(1-\frac{x}{h}\right)\left(1-\frac{y}{h}\right), \quad \phi^{(2)}(x,y) = \frac{x}{h}\left(1-\frac{y}{h}\right),$$
$$\phi^{(3)}(x,y) = \left(1-\frac{x}{h}\right)\frac{y}{h}, \quad \phi^{(4)}(x,y) = \frac{xy}{h^2}$$



- The Lagrange basis of the finite element space is given as

$$V_h := \{\phi_h \in C(\bar{\Omega}) \mid \phi\big|_T \in Q^1 := \operatorname{span}\left(\phi_h^{(1)}, \phi_h^{(2)}, \phi_h^{(3)}, \phi_h^{(4)}\right)\}$$

- The **Lagrange basis** or **nodal basis** is given by

$$\phi_h^{(i)} \in V_h : \quad \phi_h^{(i)}\big|_T \in Q^1, \quad \phi_h^{(i)}(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Step 3-Discretized formulation

Poisson equation

- Starting point: weak formulation of Laplace equation

$$u \in \mathcal{V} \quad (\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in \mathcal{V}$$

conforming finite element

- We discretize the **trial functions** $u_h \in V_h \subset \mathcal{V}$ and the **test functions** $\phi_h \in V_h \subset \mathcal{V}$

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h.$$

Step 4-Linear systems of equations

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h. \tag{1}$$

- The finite element space is given by a local basis

finite dimensional

$$V_h = \text{span}\{\phi_h^{(1)}, \dots, \phi_h^{(N)}\}$$

- We split (~~??~~ 1) into N equations

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h^{(i)}) = (f, \phi_h^{(i)}) \quad \forall i = 1, \dots, N \tag{2}$$

- We write the unknown solution $u_h \in V_h$ as

scalar values

$$u_h(x,y) = \sum_{j=1}^{N} \mathbf{u}_j \phi_h^{(j)}(x,y)$$

and insert this notation into (**??**)

$$\sum_{j=1}^{N} \underbrace{(\nabla \phi_h^{(j)}, \nabla \phi_h^{(i)})}_{A_{ij}} \overset{\text{Scalar}}{\mathbf{u}_j} = \underbrace{(f, \phi_h^{(i)})}_{f_i} \quad \forall i = 1, \dots, N \tag{3}$$

- This is equivalent to a **linear system of equations**

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \quad \mathbf{A}_{ij} := (\nabla \phi_h^{(j)}, \nabla \phi_h^{(i)}), \quad \mathbf{f}_i := (f, \phi_h^{(i)})$$

Outline

- Step 1: Approximation estimates (qualitative; no convergence rates in terms of h powers yet)
- Step 2: Interpolation estimates (yielding local h powers)
- Step 3: Convergence results (yielding global h powers)

We have

Poisson Problem

analytic $\leftarrow (\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in V,$

discretisation $(\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h.$

Taking in particular only discrete test functions from $V_h \subset V$ (**confroming finite elements**) and subtraction of both equations yields:

Proposition 1 (Galerkin orthogonality). It holds:

$a(u,v) = (\nabla u, \nabla v)$

$$(\nabla(\underbrace{u - u_h}_{\text{error}}), \phi_h) = 0 \quad \forall \phi_h \in V_h,$$

$e_h = u - u_h$

or in the more general notation:

$$a(u - u_h, \phi_h) = 0 \quad \forall \phi_h \in V_h.$$

Proof:

$a(u, \phi) = (f, \phi) \quad \forall \phi \in V$. conforming F.E

$V_h \subset V$

$$\text{I} \quad a(u, \phi_h) = (f, \phi_h) \qquad \text{Choose } \phi = \phi_h$$

$$\text{I} - \text{II} \quad \text{II} \quad a(u_h, \phi_h) = (f, \phi_h)$$

$$a(u, \phi_h) = 0 \implies$$

Poisson eq.

$$(\nabla(u - u_h), \nabla \phi_h) = 0$$

Numerical analysis: Galerkin orthogonality: illustration

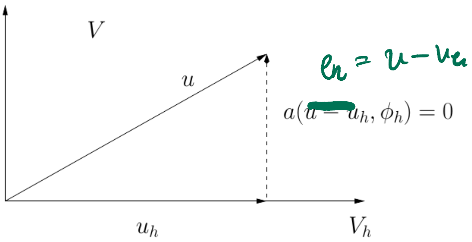


Figure: Illustration of Galerin orthogonality.

Proof. Taking $\phi_h \in V_h$ in both previous equations yields:

$$(\nabla u, \nabla \phi) - (\nabla u_h, \nabla \phi_h) = (f, \phi) - (f, \phi_h).$$

Taking both equations in the discrete space V_h means $\phi := \phi_h$ (is no problem since $V_h \subset V$) and with that

$$(f, \phi_h) - (f, \phi_h) = 0.$$

$\square$

Proposition 2. For the Poisson equation the best approximation holds true

$$\|u - u_h\|_V = \min_{\phi_h \in V_h} \|u - \phi_h\|_V$$

Proof:

$$\|\nabla(u-u_h)\|^2 = a(u-u_h, u-u_h) = a(u-u_h, u \pm \phi_h - u_h)$$

$$= a(u-u_h, u-\phi_h) + \underbrace{a(u-u_h, \phi_h - u_h)}_{=0 \text{ by G.O.}}$$

$$= a(u-u_h, u-\phi_h)$$

$$= (\nabla(u-u_h), \nabla(u-\phi_h)) \quad \leftarrow \text{Cauchy-Schwarz ineq}$$

$$= \cancel{\|\nabla(u-u_h)\|} \cdot \|\nabla(u-\phi_h)\|$$

$$\Leftrightarrow \|\nabla(u-u_h)\| \le \|\nabla(u-\phi_h)\|$$

V_h

$$\|\nabla(u-u_h)\| = \min_{\phi_h \in V_h} \|\nabla(u-\phi_h)\|$$

local estimates on T

Numerical analysis: Step 2. Interpolation estimates

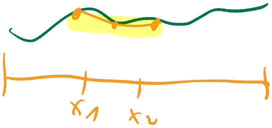
First, we need to construct an interpolation operator in order to approximate the continuous solution at certain nodes.

Definition 3 (Interpolation operator)**.** Let $\Omega = (0,1)^2$. A P_1 interpolation operator $i_h : C^1(\Omega) \to V_h$ is defined by

$$(i_h v)(x) = \sum_{j=0}^{n+1} v(x_j)\phi_j(x) \quad \forall v \in H^1.$$

This definition is well-defined since H^1 functions are continuous in 2D and are pointwise defined. The interpolation i_h creates a piece-wise linear function that coincides in the support points x_j with its H^1 function.

1D example



local estimates

Lemma 4. For a function $u \in H^2$, it exists a constant C (independent of h) such that

$$\|u - i_h u\|_{L^2(T)} \leq Ch^2 \|(\nabla)^2 u\|_{L^2(T)},$$
$$|u - i_h u|_{H^1(T)} \leq Ch \|(\nabla)^2 u\|_{L^2(T)}.$$

Lemma 5. There exists a constant C (independent of h) such that for all $u \in H^1(\Omega)$, it holds

$$\|i_h u\|_{H^1} \leq C\|u\|_{H^1(T)}$$

and

$$\|u - i_h u\|_{L^2(T)} \leq Ch|u|_{H^1(T)}.$$

Theorem 6. Let $u \in H_0^1$ and $u_h \in V_h$ be the solutions of the continuous and discrete Poisson problems. Then, if $u \in H^2$ (for instance when $f \in L^2$ and in higher dimensions when the domain is sufficiently smooth or polygonal and convex), we have

$$\boxed{\|u - u_h\|_{H^1(\Omega)} \leq Ch\|(\nabla)^2 u\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}.}$$

Thus the convergence in the H^1 norm (the energy norm) is linear and depends continuously on the problem data.

error

Level	Elements	DoFs	h	L2 err	H1 err
2	16	25	1.11072	0.0955104	0.510388
3	64	81	0.55536	0.0238811	0.252645
4	256	289	0.27768	0.00597095	0.126015
5	1024	1089	0.13884	0.00149279	0.0629697
6	4096	4225	0.06942	0.0003732	0.0314801
7	16384	16641	0.03471	9.33001e-05	0.0157395
8	65536	66049	0.017355	2.3325e-05	0.00786965
9	262144	263169	0.00867751	5.83126e-06	0.00393482
10	1048576	1050625	0.00433875	1.45782e-06	0.00196741
11	4194304	4198401	0.00216938	3.64448e-07	0.000983703

:2

- The elements are $K_i, i = 0, \ldots, n$
- The DOFs represent the number of nodal points $x_i, i = 0, \ldots, n+1$

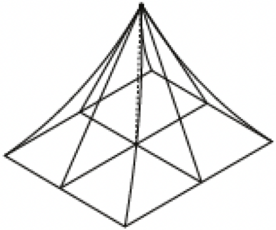
Assembling the matrix

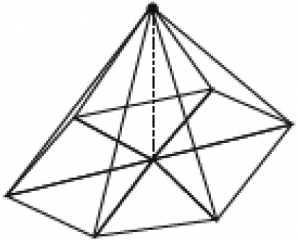
- We must compute the matrix entries

$$A_{ij} = (\nabla\phi_h^{(j)}, \nabla\phi_h^{(i)}) = \int_\Omega \nabla\phi_h^{(j)} \cdot \nabla\phi_h^{(i)}\,\mathrm{d}x = \sum_{T\in\Omega_h} \int_T \nabla\phi_h^{(j)} \cdot \nabla\phi_h^{(i)}\,\mathrm{d}x$$

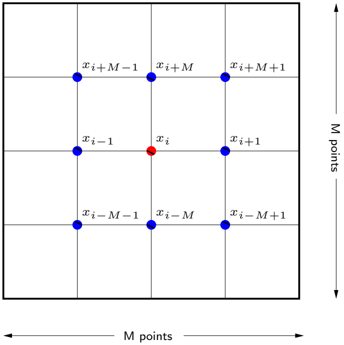
- Each **nodal basis function** $\phi_h^{(i)}$ is non-zero only in the four quadrilaterals touching x_i
- The product $\nabla\phi_h^{(j)} \cdot \nabla\phi_h^{(i)}$ is non-zero only on elements T that have both points x_i and x_j in common

bi-linear basis functions





Bi-linear finite elements for Laplace



- Regular mesh with $N = M \cdot M$ nodes
- The test function $\phi_h^{(i)}$ couples with itself and 8 further testfunctions
- The matrix elements A_{ij} must only be computed in 4 elements

- We first compute all couplings in every element T_k for $k = 1, 2, 3, 4$

$$a_{ij}^T := \int_T \nabla \phi_h^{(i)} \cdot \nabla \phi_h^{(j)} \,\mathrm{d}x$$

- Then, we put it all together in the global matrix

linear finite elements in 2D

- We first discretize the domain, then we set up the finite element space
- We must integrate the matrix and the right hand side

H^1**-error estimate:**

- first order for linear finite elements
- allowing for the verification of practical results

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