Time dependent partial differential equations

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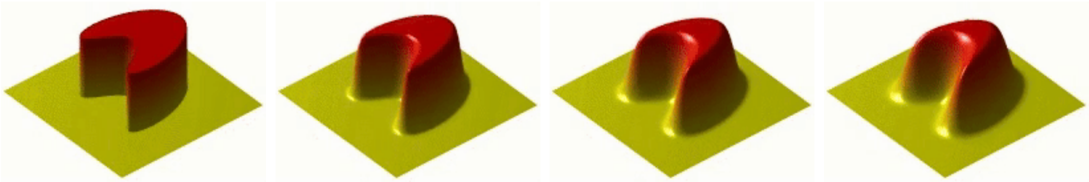
Agenda

1. The heat equation
    - Existence, uniqueness and regularity
2. The linear transport problem

The heat equation

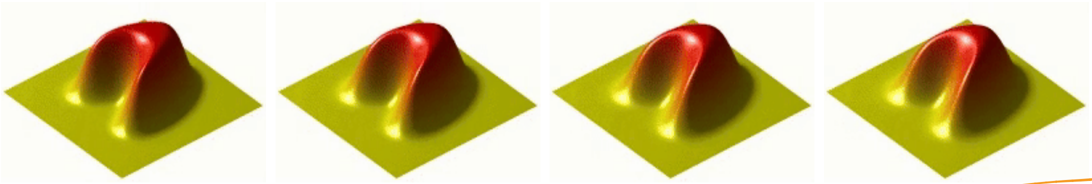


t

$$\partial_t u - \Delta u = f \text{ in } (0,T) \times \Omega$$

Initial-Boundary Value Problem

$$u(x,t) = g(x,t) \quad x \in \partial\Omega$$
$$u(x,0) = u_0(x) \quad x \in \Omega$$



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Type of equation

- The heat equation can be considered as mixture of ODE and PDE
- This concept will also be important for the numerical approximation

Variational formulation

$u \in C^1(I; C(\Omega)) \cap C(I; C^2(\Omega))$

Strong (classical) form

$$\partial_t u - \Delta u = f \text{ in } \Omega$$

Lemma Every strong solution is solution to the sapce-time variational formulation

W is a Bochner-space

$$u \in W = \{\phi \in L^2(I; H^1(\Omega)),\ \partial_t \phi \in L^2(I; \cancel{L^2(\Omega)}),\ \phi(0) = u_0\}$$

$I = [0, T]$ $\quad H^{-1}(\Omega)$

satisfying

$$((\partial_t u, \phi)) + ((\nabla u, \nabla \phi)) = ((f, \phi)) \quad \forall \phi \in W,$$

where

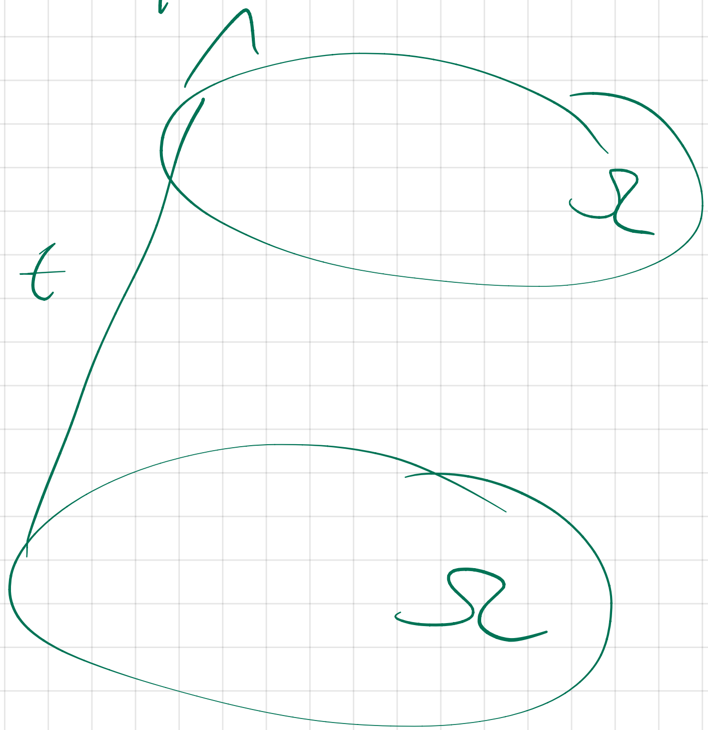
$$((u, v)) = \int_0^T \int_\Omega u(x,t) v(x,t) \, \mathrm{d}x \, \mathrm{d}t.$$

Proof

$$u_t - \Delta u = f \qquad / \cdot \varphi \in V_0$$

$$\varphi = 0 \text{ on } \partial\Omega$$

$$\iff u_t \cdot \varphi - \Delta u \cdot \varphi = f \cdot \varphi \qquad / \int_0^T \int_\Omega \cdot$$



$$\iff \int_0^T \int_\Omega u_t(x,t) \cdot \varphi(x,t) - \Delta u(x,t)\, \varphi(x,t) \, dx\, dt = \int_0^T \int_\Omega f(x,t)\, \varphi(x,t) \, dx\, dt$$

$$\iff \int_0^T \int_\Omega u_t(x,t)\, \varphi(x,t) \, dx\, dt + \int_0^T \int_\Omega \nabla u(x,t) \cdot \nabla \varphi(x,t) \, dx\, dt - \int_0^T \int_{\partial\Omega} n \cdot \nabla u(x,t) \cdot \overbrace{\varphi(x,t)}^{=0} \, dx\, dt = \int_0^T \int_\Omega f(x,t)\, \varphi(x,t) \, dx\, dt$$

Stability

Lemma Let $f \in L^2(I; L^2(\Omega))$. For every variational solution $u \in W$ to

$$((\partial_t u, \phi)) + ((\nabla u, \nabla \phi)) = ((f, \phi)) \quad \forall \phi \in W$$

it holds

$\forall T > 0$

$$\|u(T)\|^2 + \frac{1}{2} \int_0^T \|\nabla u(t)\|^2 \, \mathrm{d}t \le \|u(0)\|^2 + \frac{C_P^2}{2} \int_0^T \|f(t)\|^2 \, \mathrm{d}t$$

Proof $\quad \varphi = u$

$$((d_t u, u)) + ((\nabla u, \nabla u)) = ((f, u))$$

(1) $((\nabla u, \nabla u)) = \int_0^T \int_\Omega \nabla u \cdot \nabla u \, dx \, dt = \int_0^T \|\nabla u(t)\|_\Omega^2 \, dt$

(2) $((f, u)) = \int_0^T \int_\Omega f \cdot u \, dx \, dt = \int_0^T (f(t), u(t))_\Omega \, dt \le \int_0^T \|f(t)\| \cdot \|u(t)\| \, dt$

$$((f,u)) \le \int_0^T \|f(t)\| \cdot \|u(t)\| \, dt$$

$$\overset{\text{Poincaré}}{\le} \int_0^T \underbrace{\|f(t)\| \cdot c_p}_{a} \underbrace{\|\nabla u(t)\|}_{b} \, dt$$

$$\overset{\text{Young}}{\le} \frac{1}{2} c_p^2 \int_0^T \|f(t)\|^2 + \frac{1}{2} \int_0^T \|\nabla u(t)\|^2 dt$$

(3) $$((d_t u, u)) = \int_0^T \int_\Omega (d_t u) \cdot u \, dx \, dt$$

$$= \frac{1}{2} \int_0^T \int_\Omega d_t (u^2) \, d\, x \, dt = \frac{1}{2} \int_0^T d_t \|u(t)\|_\Omega^2 dt$$

$$= \frac{1}{2} \|u(T)\|^2 - \frac{1}{2} \|u(0)\|^2$$

$$\boxed{\begin{aligned} & d_t(u \cdot u) \\ & = d_t u \cdot u + u \cdot d_t u \\ & = 2 \cdot \partial_t u \cdot u \end{aligned}}$$

$$\Rightarrow$$

$$\frac{1}{2}\|u(T)\|^2 + \frac{1}{2}\int_0^T \|\nabla u(t)\|^2 \, dt \le \frac{1}{2}\|u(0)\|^2 + \frac{c_p^2}{2}\int \|f(t)\|^2 \, dt$$

$\square$

Uniqueness

Lemma The variational solution $u \in W$ is unique.

Proof: Assume $u_1, u_2 \in W$ are solutions

$$((d_t u_1, \varphi)) + ((\nabla u_1, \nabla \varphi)) = ((f, \varphi))$$
$$((d_t u_2, \varphi)) + ((\nabla u_2, \nabla \varphi)) = ((f, \varphi))$$

$$\Rightarrow ((d_t (\underbrace{u_1 - u_2}_{w}), \varphi)) + ((\nabla (u_1 - u_2), \nabla \varphi)) = 0$$

$$\Leftrightarrow ((d_t w, \varphi)) + ((\nabla w, \nabla \varphi)) = 0$$

With stability-estimate

$$\Rightarrow \quad \|w(T)\|^2 + \int_0^T \|\nabla w(t)\|^2 \leq \|w(0)\|^2$$

$$w(0) = u_1(0) - u_2(0) = 0$$

$$\Rightarrow \quad w = 0$$

$\square$

Smoothing property

- The heat equation has the *smoothing property:* Even if the initial solution is rough, e.g. discontinuous, the solution is smooth for every $t > 0$.

Let Ω be smooth.

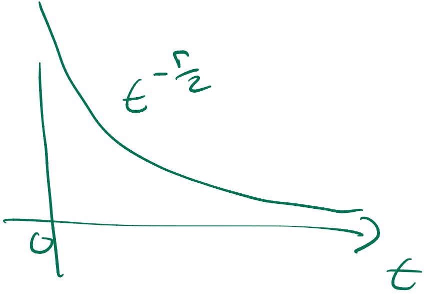
Lemma Let $u_0 \in L^2(\Omega)$ be the initial solution. For the homogenous problem

$$((\partial_t u, \phi)) + ((\nabla u, \nabla \phi)) = 0 \quad \forall \phi \in W,$$

it holds for all $r \in \mathbb{N}$

$$\|\nabla^r u(t)\| \leq c \cdot t^{-\frac{r}{2}} \cdot \|u_0\|_{L^2(\Omega)}$$

$$\xrightarrow{t \to 0} \infty$$



or: for $t > 1$

$$\|\nabla^r u(t)\| \leq c \, \|u_0\|_{L^2(\Omega)}$$

Proof (o) We know

$\tau = 0$

$$\|u(T)\|^2 + \int_0^T \|\nabla u(t)\|^2 \, dt \le \|u_0\|^2$$

$$\boxed{-\Delta = \nabla \cdot \nabla = \operatorname{div} \nabla}$$

$\tau = 1$

Idea $\varphi = -\Delta u$

$$((d_t u, -\Delta u)) + ((\nabla u, \nabla(-\Delta u))) = 0$$

$\Leftrightarrow$

$$((\nabla d_t u, \nabla u)) + \int_0^T - \int_{\partial\Omega} \overbrace{d_t u}^{=0} \cdot (n \cdot \nabla u) \, ds \, dt$$

$$+ ((-\Delta u, -\Delta u)) - \int_0^T \int_{\partial\Omega} \nabla u \cdot (n \cdot \overbrace{\Delta u}^{= d_t u = 0}) \, dx \, dt$$

$$= 0$$

$\Rightarrow$

$$((d_t \nabla u, \nabla u)) + ((-\Delta u, -\Delta u)) = 0$$

$$\Leftrightarrow \frac{1}{2}\int_0^T d_t \|\nabla u(t)\|^2 dt + \int_0^T \|\Delta u(t)\|^2 dt = 0$$

$$\Leftrightarrow \underline{\frac{1}{2}\|\nabla u(T)\|^2} + \int_0^T \|\Delta u(t)\|^2 dt = \underline{\frac{1}{2}\|\nabla u_0\|^2}$$

We showed: if $u_0 \in H^1(\Omega)$ then

$u(t) \in H^1(\Omega)$ and $u \in L^2(I; H^2(\Omega))$

But: We want to show $u(t) \in H^1(\Omega)$ for $u_0 \in L^2(\Omega)$!

Proof with "smoothing". Trick: add Factor "t"

$$\varphi = -t \cdot \Delta u$$

$$((\partial_t u, -t\,\Delta u)) + ((\nabla u, \nabla(-t\,\Delta u))) = 0$$

$$\iff ((\partial_t \nabla u, t \nabla u)) + \overbrace{((-\Delta u, -t\,\Delta u))}^{\int_0^T t\,\|\Delta u(t)\|^2\,dt} = 0$$

on the side

$$\partial_t \left(t\,|\nabla u|^2\right)$$

$$= \partial_t |\nabla u|^2 \cdot t + \|\nabla u\|^2$$

$$= 2(\nabla \partial_t u, t \cdot \nabla u) + \|\nabla u\|^2$$

$$((\partial_t \nabla u, t \nabla u)) = \frac{1}{2}\int_0^T \partial_t \left(t\,\|\nabla u\|^2\right) dt - \frac{1}{2}\int_0^T \|\nabla u(t)\|^2\,dt$$

$$= \frac{1}{2} T\,\|\nabla u(T)\|^2 - 0 - \frac{1}{2}\int_0^T \|\nabla u(t)\|^2\,dt$$

$$\frac{T}{2}\|\nabla u(T)\|^2 + \int_0^T t\cdot\|\Delta u(t)\|^2\,dt = 0 + \frac{1}{2}\int_0^T \|\nabla u(t)\|^2\,dt$$

$$\le \frac{1}{2}\|u_0\|^2$$

$$\Rightarrow \quad \ge c_s\|u\|_{H^2} \quad ?$$

$$r=1 \qquad \|\nabla u(T)\|^2 + \frac{2}{T}\int_0^T t\,\underbrace{\|\Delta u(t)\|^2}_{\le c\cdot\|u\|_{H^2}}\,dt \le \frac{1}{T}\|u_0\|^2$$

General $\qquad \varphi = (-\Delta u)^r \cdot t^r \qquad r = 0, 1, 2, \ldots$

<u>General Proof</u>

$$((d_t u, \varphi)) + ((\nabla u, \nabla \varphi)) = 0$$

$$\varphi = t^r(-\Delta^r u)$$

$$((d_t u, t^r(-\Delta)^r u)) + ((\nabla u, t^r(-\Delta)^r u)) = 0$$

r-times partial Integration

$$(-\Delta)^r = (-\mathrm{div}\, \nabla)^r$$

$$\Longleftrightarrow$$

$$\underbrace{((d_t \nabla^r u, t^r \nabla^r u))}_{= \frac{1}{2}\int_0^T d_t\left(t^r \|\nabla^r u\|^2\right) dt \;-\; \frac{1}{2}\int_0^T r \cdot t^{r-1} \|\nabla^r u\|^2 dt} + \underbrace{((\nabla^{r+1} u, t^r \nabla^{r+1} u))}_{= \int_0^T t^r \|\nabla^{r+1} u\|^2 dt} = 0$$

$$\Longleftrightarrow \underbrace{\frac{1}{2}\int_0^T d_t\left(t^r \|\nabla^r u\|^2\right)dt + \int_0^T t^r \|\nabla^{r+1} u\|^2 dt}_{} \le \frac{r}{2}\int_0^T t^{r-1}\|\nabla^r u(t)\|^2 dt$$

$$= \frac{1}{2} T^r \|\nabla^r u(T)\|^2 - 0$$

(*)

$$\Longleftrightarrow T^r\|\nabla^r u(T)\|^2 + 2\int_0^T \underline{t^r \|\nabla^{r+1} u\|^2 dt} \le r \cdot \int_0^T t^{r-1}\|\nabla^r u(t)\|^2 dt$$

$\int_0^T t^{r-1}\|\nabla^r u(t)\|^2 dt$ is estimated using (*) for $r-1$

$$\Rightarrow \int_0^T t^{r-1}\|\nabla^r u(t)\|^2 dt \le c \cdot \int_0^T t^{r-2}\|\nabla^{r-1} u\|^2 dt \le \dots \le c \cdot \int_0^T \|\nabla u\|^2 dt \le c\|u_0\|^2 \qquad \blacksquare$$