Finite Elements

Thomas Richter, Otto von Guericke University of Magdeburg

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Otto the Great - King (from 936) and Emporor (from 962) who liked Magdeburg

Cathedral of Magdeburg (built 1209 - 1520)

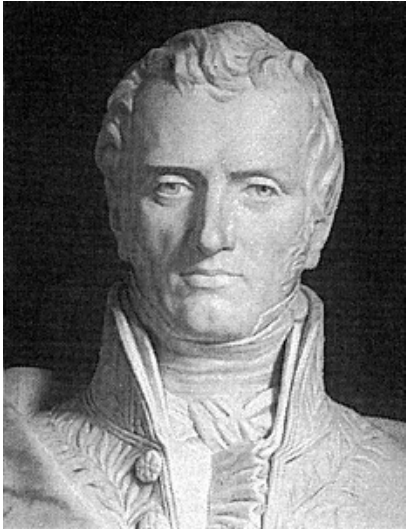
Last project of Friedensreich Hundertwasser

Inventor Otto von Guericke (pressure)

Magdeburg in Germany

Agenda

1. Introduction to PDE's
    - Models
    - Types of linear PDE's
2. The Poisson Equation
    - Formulations
    - Existence and regularity of solutions



$$\operatorname{div} v = 0$$

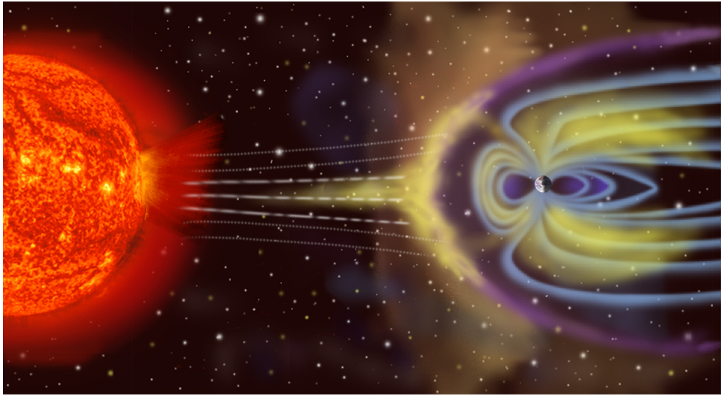
$$\rho\Big(\partial v + (v \cdot \nabla)v\Big) - \rho\nu\Delta v + \nabla p = \rho f$$

Velocity vector v and pressure p are the unknowns

Density ρ and viscosity ν are parameters

G.P. Galdi: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, **Springer, 2011**
V. John: *Finite Element Methods for Incompressible Flow Problems*, **Springer, 2016**

Maxwell's equation for electromagnetism



Source: Wikipedia

$$
\begin{aligned}
\nabla \cdot E &= \frac{\rho}{\epsilon} \\
\nabla \cdot B &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \times B &= \mu_0\left(J + \epsilon\frac{\partial E}{\partial t}\right)
\end{aligned}
\tag{1}
$$

E is the electric field, B is the magnetic field.

A PDE system to model infectious disease spread

$$
\begin{aligned}
\frac{\partial I}{\partial t} + \nabla \cdot (\mathbf{u} I) + \nabla_\ell \cdot (\mathbf{G} I) + C I = F \quad & \text{in} \quad (0, T_\infty] \times \Omega_x \times \Omega_\ell, \\
I(t, \mathbf{x}, \ell) = g_n \quad & \text{in} \quad (0, T_\infty] \times \partial\Omega_x^- \times \Omega_\ell, \\
I(t, \mathbf{x}, (\ell_v, 0, \ell_a)) = B_{\text{nuc}} \quad & \text{in} \quad (0, T_\infty] \times \Omega_x \times L_v \times L_a, \\
I(t, \mathbf{x}, (0, \ell_d > 0, \ell_a)) = 0 \quad & \text{in} \quad (0, T_\infty] \times \Omega_x \times L_d \times L_a, \\
I(0, \mathbf{x}, \ell) = I_0 \quad & \text{in} \quad \Omega_x \times \Omega_\ell.
\end{aligned}
$$

I is the population with $I = I(x, l, t)$ where t is the time, $x \in \Omega$ is the space and $l \in \Omega_l$ is a property (e.g. severity, duration of the infection and age of the population)

Sashikumaar Ganesan, Deepak Subramani: *Spatio-temporal predictive modeling framework for infectious disease spread*, **Scientific Reports 2021**

Common structure

Definition A *partial differential equation* is a differential equation which computes an unkown function $u : \mathbb{R}^d \to \mathbb{R}^n$ for $d > 1$ where various partial derivatives interact.

- **Navier-Stokes** Velocity

$$v = v(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$$

  and pressure

$$p = p(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$$

Common structure

The **nabla operator** ∇

$$f : \mathbb{R}^d \to \mathbb{R} \qquad \nabla f(x) = \begin{pmatrix} \partial_1 f(x) \\ \partial_2 f(x) \\ \vdots \\ \partial_d f(x) \end{pmatrix}$$

$$F : \mathbb{R}^d \to \mathbb{R}^d \qquad \nabla F(x) = \begin{pmatrix} \partial_1 F_1(x) & \partial_2 F_1(x) & \cdots & \partial_d F_1 f(x) \\ \partial_1 F_2(x) & \partial_2 F_2(x) & \cdots & \partial_d F_2 f(x) \\ \vdots & & \ddots & \vdots \\ \partial_1 F_d(x) & \partial_2 F_d(x) & \cdots & \partial_d F_d f(x) \end{pmatrix}, \qquad \begin{aligned} \nabla \cdot F &= \mathrm{div}\ F \\ &= \partial_1 F_1(x) + \partial_2 F_2(x) + \cdots + \partial_d F_d(x) \end{aligned}$$

$$f : \mathbb{R}^d \to \mathbb{R} \qquad \Delta f = \nabla \cdot \nabla f(x) = \partial_{11} f(x) + \partial_{22} f(x) + \cdots + \partial_{dd} f(x)$$

Three Types of linear PDEs

Poisson / Laplace equation A stationary equation with no distinct direction

$$\boxed{-\Delta u(x) = f(x) \text{ in } \Omega} \qquad u = g \text{ on } \partial\Omega \qquad \text{Elliptic PDE}$$

One of the main ingredients in many differential equation models. Usually describing diffusive effects (at infinite extension speed)

Heat equation A nonstationary problem with no distinct spatial direction

$$\boxed{\partial_t u(x,t) - \Delta u(x,t) = f(x,t) \text{ in } \Omega} \qquad u = g \text{ on } \partial\Omega, \quad u = u_0 \text{ for } t = 0 \qquad \text{Parabolic PDE}$$

Describing slow diffusion, e.g. of heat

Wave euation / Advection equation A nonstationary problem with a spatial direction of transport

$$\boxed{\partial_{tt} u(x,t) - \Delta u(x,t) = f(x,t) \text{ in } \Omega} \qquad u = g \text{ on } \partial\Omega, \quad u = u_0,\ \partial_t u = u_1 \text{ for } t = 0 \qquad \text{Hyperbolic PDE}$$

Describes the spatio-temporal dynamics of waves.

$$\boxed{\partial_t u(x,t) + \vec{\beta}(x,t) \cdot \nabla u(x,t) = f(x,t) \text{ in } \Omega} \qquad u = g \text{ on } \partial\Omega_{in}, \quad u = u_0, \text{ for } t = 0 \qquad \text{Hyperbolic PDE}$$

Describes the transport in a velocity field $\vec{\beta}(x,t)$

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The Poisson Equation

The **classical** or **strong formulation** of the Poisson problem on $\Omega \subset \mathbb{R}^d$

$$u \in C^2(\Omega) \cap C(\bar{\Omega}) \qquad -\Delta u(x) = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{2}$$

$-\partial_{xx} u - \partial_{yy} u = f$

Lemma (variational problem) Each solution to (2) is also solution to the **variational problem**

$$u \in V := \{\phi \in C^1(\Omega) \cap C(\bar{\Omega}) : \quad \phi = 0 \text{ on } \partial\Omega\} \quad \int_\Omega \nabla u(x) \cdot \nabla \phi(x)\, \mathrm{d}x = \int_\Omega f(x)\phi(x)\, \mathrm{d}x \quad \forall \phi \in V \tag{3}$$

The variational formulation (3) is equivalent to the **minimization problem**

$$u \in V := \{\phi \in C^1(\Omega) \cap C(\bar{\Omega}) : \quad E(u) \leq E(\phi) := \frac{1}{2}\int_\Omega |\nabla \phi(x)|^2\, \mathrm{d}x - \int_\Omega f(x)\phi(x)\, \mathrm{d}x \quad \forall \phi \in V \tag{4}$$

If a solution $u \in V$ satisfies the regularity $u \in C^2(\Omega)$ it is also solution to (2)

Proof ...

$(2) \Rightarrow (3)$

Proof

let $u \in C^2(\Omega) \cap C(\bar{\Omega})$

(2) $-\Delta u = f \quad / \cdot \varphi \in V_0 = \{ v \in C^1, \ v = 0 \text{ on } \partial\Omega \}$

$\Updownarrow$

$-\Delta u \cdot \varphi = f \cdot \varphi \quad / \int$ " – "

Fundamental Thm of variations

$\Updownarrow$

$-\int_\Omega \Delta u \cdot \varphi \, dx = \int_\Omega f \cdot \varphi \, dx$

$\Updownarrow$

$\int_\Omega \nabla u \cdot \nabla \varphi \, dx - \cancel{\int_{\partial\Omega} n \cdot \nabla u \cdot \varphi \, ds} = \int_\Omega f \cdot \varphi \, dx \quad (3)$

$= 0$

$u \in V_0$ is enough

(3) $\Rightarrow$ (4)

We find the <u>min</u> of $E(\varphi) = \frac{1}{2}\|\nabla\varphi\|^2 - (f,\varphi)$

$$E(u+\varphi) - E(u) = \frac{1}{2}\|\nabla(u+\varphi)\|^2 - \frac{1}{2}\|\nabla u\|^2 + (f, u+\varphi) - (f,\varphi)$$

$$= \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla\varphi\|^2 + (\nabla u, \nabla\varphi) \quad - \frac{1}{2}\|\nabla u\|^2 + (f,\varphi)$$

$$= \underbrace{(\nabla u, \nabla\varphi) - (f,\varphi)} + \frac{1}{2}\|\nabla\varphi\|^2$$

$$= \frac{1}{2}\|\nabla\varphi\|^2 \geq 0$$

u solves (3) $\Rightarrow$

$(\nabla u, \nabla\varphi) = (f,\varphi)$ $\Rightarrow$ $E(u+\varphi) \geq E(u)$ $\forall\varphi$

$\Rightarrow$ u is min (4)

(4) $\Rightarrow$ (3)

u is Min. $\Rightarrow$ $\forall \varphi \in V_0$ $\forall t \in \mathbb{R}$

$$E(u + t\varphi) - E(u) \geq 0$$

previous Slide:

$$E(u+t\varphi) - E(u) = \frac{t^2}{2} \|\nabla \varphi\|^2 + t \cdot \big((\nabla u, \nabla \varphi) - (f, \varphi) \big) \geq 0$$

$$t > 0 \Rightarrow \quad (\nabla u, \nabla \varphi) - (f, \varphi) \geq \overbrace{-\frac{t}{2} \|\nabla \varphi\|^2}^{\leq 0}$$

$$t < 0 \Rightarrow \quad (\nabla u, \nabla \varphi) - (f, \varphi) \leq \underbrace{-\frac{t}{2} \|\nabla \varphi\|^2}_{\geq 0}$$

$\Rightarrow$

$$-\frac{|t|}{2} \|\nabla \varphi\|^2 \leq (\nabla u, \nabla \varphi) - (f, \varphi) \leq \frac{|t|}{2} \|\nabla \varphi\|^2$$

$t \to 0 \Rightarrow (\nabla u, \nabla \varphi) = (f, \varphi)$ (3)

$\blacksquare$

$$u \in V := \{\phi \in C^1(\Omega) \cap C(\bar{\Omega}) : \quad E(u) \leq E(\phi) := \frac{1}{2} \int_\Omega |\nabla \phi(x)|^2 \,\mathrm{d}x - \int_\Omega f(x)\phi(x) \,\mathrm{d}x \quad \forall \phi \in V \tag{5}$$

Lemma (Existence)

If the domain Ω has sufficient regularity, there is a unique solution to every $f \in L^2(\Omega)$. It holds

$$u \in H_0^1(\Omega) : \quad \{\phi \in L^2(\Omega), \text{ it exists a Cauchy-Sequence } u_k \in V \cap L^2(\Omega) \text{ with } \|u - u_k\|_{L^2} \to 0$$
$$\text{such that} \|\nabla(u_k - u_l)\| \to 0\}$$

Proof ...

$$(u, v) := \int_\Omega u(x)v(x) \,\mathrm{d}x, \quad (\nabla u, \nabla v) := \int_\Omega \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x.$$

$u \in V_0$ $\quad E(u) \leq E(\varphi) = \frac{1}{2}\int_\Omega |\nabla\varphi|^2 dx - \int_\Omega f\cdot\varphi \, dx$ $\quad \forall \varphi \in V_0$

(i) Show $E(\varphi) > -\infty$

$$E(\varphi) = \frac{1}{2}\|\nabla\varphi\|^2 - (f,\varphi)$$

$$\geq \frac{1}{2}\|\nabla\varphi\|^2 - \|f\|\cdot\|\varphi\|$$

$$\geq \frac{1}{2}\|\nabla\varphi\|^2 - \overbrace{c_p\|f\|}^{a}\cdot\overbrace{\|\nabla\varphi\|}^{b}$$

$\varepsilon = 1$

$$\geq \frac{1}{2}\|\nabla\varphi\|^2 - \frac{1}{2}\|\nabla\varphi\|^2 - \frac{c_p^2}{2}\|f\|^2$$

$$= -\frac{c_p^2}{2}\|f\|^2 > -\infty$$

Notation

$$(f,\varphi) = \int_\Omega f\cdot\varphi \, dx$$

Poincare-Inequality

$\forall u \in V_0$ it holds

$$\|u\| \leq c_p \|\nabla u\|$$

Young's inequality

$a, b \in \mathbb{R}$, $\varepsilon > 0$

$$a\cdot b \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$$

$$d = \inf_{\varphi \in V_0} E(\varphi) > -\infty$$

(11) There is a sequence $E(u_k)$ $k \in \mathbb{N}$

such that $E(u_k) \xrightarrow[k \to \infty]{} d$

Show that u_k is a Cauchy-sequence

$\|\nabla(u_k - u_l)\|^2 \to 0$ for $k, l \to \infty$

Idea $E(u_k - u_l) = \frac{1}{2}\|\nabla(u_k - u_l)\|$

$- (f, u_k - u_l)$

u_k is Cauchy

$$\|\nabla(u_k - u_l)\|^2 = 2\|\nabla u_k\|^2 + 2\|\nabla u_l\|^2 - \|\nabla(u_k+u_l)\|^2$$

Parallelogram-Identity

$$= 2\|\nabla u_k\|^2 + 2\|\nabla u_l\|^2 - 4\left\|\nabla\left(\frac{u_k+u_l}{2}\right)\right\|^2$$

$$= 4E(u_k) + 4(f, u_k) + 4E(u_l) + 4(f, u_l) - 8E\left(\frac{u_k+u_l}{2}\right) - 8\left(f, \frac{u_k+u_l}{2}\right)$$

$= 0$

$$= 4E(u_k) + 4E(u_l) \underbrace{-8\underbrace{E\left(\frac{u_k+u_l}{2}\right)}_{\geq d}}_{\leq -8d} \xrightarrow[k,l\to\infty]{} \leq 0$$

$4E(u_k) \xrightarrow{k\to\infty} 4d$, $\quad 4E(u_l) \xrightarrow{l\to\infty} 4d$

$\Rightarrow u_k$ Cauchy

(3)

$\Rightarrow$ there exists a limit $u_k \to u$

with $u \in H_0^1(\Omega) =$

$\{ \varphi \in L^2$: there exists a Cauchy sequence $\varphi_k \in V_0$ with $\|\nabla(\varphi - \varphi_k)\| \to 0 \}$

Boundary conditions

Dirichlet Problem Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$$

Neumann Problem Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega, \quad \partial_n u = g \text{ on } \partial\Omega$$

Robin Problem Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega, \quad \alpha u + \partial_n u = g \text{ on } \partial\Omega, \quad \alpha > 0$$

Non-homogenous Dirichlet Problem - A trick for numerics

Find $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$$

Construct $u_D \in H^1(\Omega)$ such that

$$u_D = g \text{ on } \partial\Omega$$

Split the solution

$$u = u_D + u_0, \quad u_0 \in H_0^1(\Omega), \quad (\nabla u, \nabla \phi) = (\nabla u_D, \nabla \phi) + (\nabla u_0, \nabla \phi).$$

Non-homogenous Dirichlet problem Find $u_0 \in H_0^1(\Omega)$

$$(\nabla u_0, \nabla \phi) = (f, \phi) - (\nabla u_D, \nabla \phi) \quad \forall \phi \in H_0^1(\Omega)$$

- Numerically, this trick will be used in the Python classes in the afternoon
- Theoretical problem: *Where do we get the extension of the boundary values g to an H^1-function u_D?*

Wloka: *Partial Differential Equations*, **Cambridge University Press, 1987**
Rudin: *Functional Analysis*, **McGraw-Hill, 1991**

!Warning! The function space H^1 is special

- H^1 functions must not have a "classical derivative"
- Example: absolute value

$$|x| \in H^1(I), \quad I = (-1, 1)$$

- In $\mathbb{R}$, H^1 functions are continuous
- In $\mathbb{R}^d$ for $d \geq 2$ this is not necessarily true

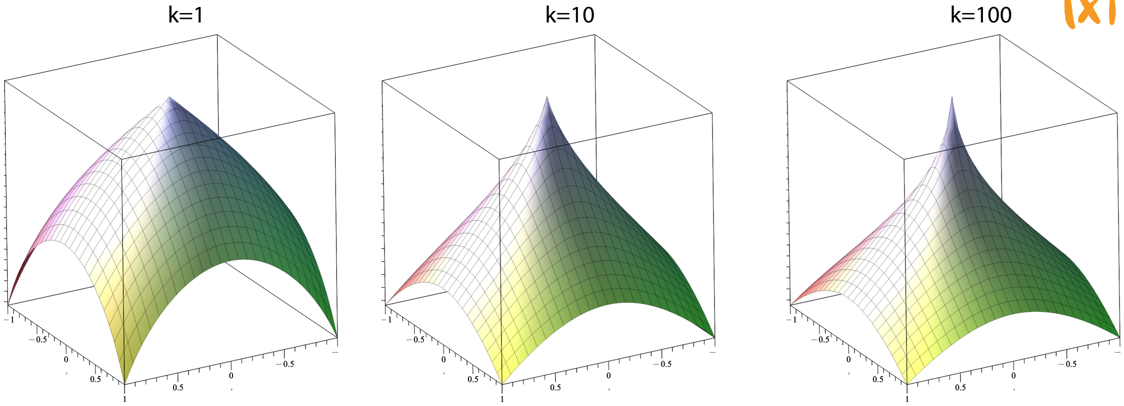
Regularity of solutions

$$C^\infty(\Omega) \ni v_k = \log\left(\log\left(\frac{1}{|x| + \frac{1}{k}}\right) + 1\right) \to \log\left(\log\left(\frac{1}{|x|}\right) + 1\right) =: v \in H^1_0(\Omega)$$

It holds $v(x) \to \infty$ for $x \to 0$ but

$$\int_{x^2+y^2<1} |\nabla v(x,y)|^2 \, \mathrm{d}x\mathrm{d}y = 2\pi$$

$\Omega = \{x \in \mathbb{R}^2, |x|^2 < 1\}$



Definition (Sobolev Spaces) We define the function spaces

$$H^m(\Omega) = \{\phi \in L^2(\Omega) \quad \|\nabla^k \phi\|_{L^2(\Omega)} < \infty \; k = 1, \ldots, m\}$$

Lemma (Regularity) If

$$f \in H^m(\Omega)$$

and if the domain Ω has a boundary that has the regularity C^{k+2}, the solution has the regularity

$$u \in H^{m+2}(\Omega)$$

Special case: If $f \in L^2(\Omega)$ and Ω is a polygonal and convex domain it holds

$$u \in H^2(\Omega)$$

and

$$\|\nabla^2 u\| \leq c_s \|\Delta u\| = \|f\|$$