## Navier-Stokes

$$\left.\begin{aligned} -\frac{1}{Re}\Delta v + \overbrace{(v\cdot\nabla)v}^{\text{"convective"}} + \nabla p &= f \\ \nabla\cdot v &= 0 \end{aligned}\right\} \quad \text{Nonlinear Eq.!}$$

## Weak Form.

$v \in H_0^1(\Omega)$

$p \in L^2(\Omega)$

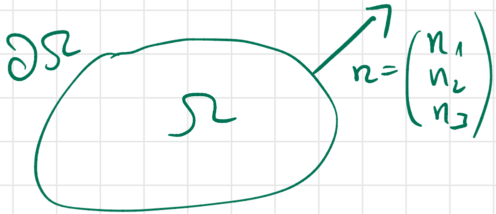
$$\frac{1}{Re}(\nabla v, \nabla\varphi) - (p, \nabla\cdot\varphi) + ((v\cdot\nabla)v, \varphi) = (f, \varphi) \qquad \varphi \in H_0^1(\Omega)$$

$$(\nabla\cdot v, \xi) = 0 \qquad \xi \in L^2(\Omega)$$

**Lemma** For $v, w \in H_0^1(\Omega)$ with $\operatorname{div} v = 0$, it holds

$$((v\cdot\nabla)w, w) = 0$$

**Proof** $\quad ((v\cdot\nabla)w, w) = \int_\Omega \sum_{i=1}^{d} ((v\cdot\nabla)w)_i \cdot w_i \, dx$




$$= \int_\Omega \sum_{i=1}^{d} \sum_{j=1}^{d} v_j \partial_j w_i \cdot w_i \, dx$$

$$= \sum_{\substack{i=1\\ j=1}}^{d} -\int_\Omega w_i \underbrace{\partial_j}_{x_j} (v_j w_i)\, dx + \int_{\partial\Omega} \underbrace{w_i v_j w_i n_j}_{=0} \, ds$$

$$= \sum_{i,j=1}^{d} -\int w_i \partial_j v_j \, w_i + w_i v_j \partial_j w_i \, dx$$

$$= -\sum_{i=1}^{d} \int \operatorname{div} v \cdot w_i^2 \, ds - \sum_i \int w_i \cdot ((v\cdot\nabla)w)_i \, ds$$

$$= -\int_\Omega (\operatorname{div} v)\, |w|^2 \, dx \; - \underbrace{\big(w, (v\cdot\nabla) w\big)}_{\substack{\| \\ -\left((v\cdot\nabla) w,\, w\right)}}$$

$$\Downarrow$$

$$2\Big(\big((v\cdot\nabla) w, w\big)\Big) = -\int_\Omega \underbrace{(\operatorname{div} v)}_{=0} \, \|w\|^2 \, dx = 0$$

Important: $\operatorname{div} v = 0$

$v = 0$ on $\partial\Omega$

or $w = 0$ on $\partial\Omega$

**Lemma:** Let $v \in H_0^1(\Omega)$, $p \in L^2(\Omega)$ be a solution to the Navier-Stokes-Eq.

Then:

$$\|\nabla v\| \leq Re \, \|f\|_{-1}$$

"Energy Estimate"

$$\left( \text{or} \quad \|\nabla v\| \leq Re \cdot c_p \, \|f\| \right)$$

**Proof**

$$\frac{1}{Re}(\nabla v, \nabla \varphi) - (p, \nabla \cdot \varphi) + ((v \cdot \nabla) v, \varphi) = (f, \varphi) \quad \varphi \in H_0^1(\Omega)$$

$$(\nabla \cdot v, \xi) = 0 \quad \xi \in L^2(\Omega)$$

$$\left.\begin{aligned} \varphi &= v \\ \xi &= p \end{aligned}\right\} \Rightarrow \frac{1}{Re}(\nabla v, \nabla v) - \underbrace{(p, \nabla \cdot v)}_{=0} + \underbrace{((v \cdot \nabla) v, v)}_{=0 \text{ if } v \in H_0^1(\Omega)} = (f, v)$$

$$(\nabla \cdot v, p) = 0$$

$$\Longleftrightarrow$$

$$\frac{1}{Re}\|\nabla v\|^2 = (f, v) \le \|f\| \cdot \|v\| \le c_p \|f\| \cdot \|\nabla v\| \quad \cdot \frac{Re}{\|\nabla v\|}$$

$$\Downarrow$$

$$\|\nabla v\| \le c_p Re \|f\|$$

$\square$

**Theorem** For every $f \in H^{-1}(\Omega)$ there exists a solution to the stationary Navier-Stokes equations. It holds

$$\|\nabla v\| + \|p\| \le c \cdot \|f\|_{-1}$$

where $c$ depends on $Re$ and the inf-sup constant $\gamma$

$$\gamma \|p\| \le \sup_{\varphi \in H_0^1(\Omega)} \frac{(p, \nabla \cdot \varphi)}{\|\nabla \varphi\|}$$

The solution is unique, if

$$c^2 Re^2 \|f\|_{-1} \le 1$$

in Applications $Re \approx 10^6 \sim 10^9$

## Idea of the proof

remove pressure

(I) Restrict Eq. to velocity $v \in V_0 := \{\varphi \in H_0^1(\Omega),\ \operatorname{div} \varphi = 0\}$

$$v \in V_0 : \frac{1}{Re}(\nabla v, \nabla \varphi) + ((v \cdot \nabla) v, \varphi) = (f, \varphi) \quad \forall \varphi \in V_0$$

discretization

(II) Galerkin-Approach: We construct a sequence of finite dimensional spaces

$$V_m = \operatorname{span}\{w_1, w_2, \dots, w_m\}$$

such that $\displaystyle\bigcup_{m=1}^{\infty} V_m = V_0$

(III) For $m \geq 1$ Find $v_m \in V_m$, $v_m = \sum_{i=1}^{m} \alpha_m^i w_i$

$$\frac{1}{Re}(\nabla v_m, \nabla \varphi_m) + (v_m \cdot \nabla v_m, \varphi_m) = (f, \varphi_m) \quad \forall \varphi_m \in V_m$$

nonlinear but finite dim. Problem in $\mathbb{R}^m$

(iv) (page 89-90) Initial $v_m^{(0)} = 0$ (or better guess)

define Iteration: $v_m^{(\ell+1)} = Q_m(v_m^{(\ell)})$

$Q_m$ Fixpoint-map defined as solution to

$$\frac{1}{Re}(\nabla Q_m(v), \nabla \varphi) + ((v \cdot \nabla) Q_m(v), \varphi) = (f, \varphi)$$

v, the old iterate is known, $Q_m(v)$ is given by a linear problem

linearization

(v) (Easy) $Q_m(v)$ exists uniquely for every $v$ and every $f$.

(vi) Show that $Q_m$ is a fixed-point mapping using

Brouwer's fixed point theorem: if $g: X \subset \mathbb{R}^n \to X$ is continuous, where $X = B_R := \{x \in \mathbb{R}^n : \|x\| \le R\}$ there exists at least one fixed point $z \in B_R$, so

$$g(z) = z$$

$$\Rightarrow \quad Q_m(v) = v$$

$\Rightarrow$ $v = Q_m(v)$ is solution

$$\frac{1}{Re}\left(\nabla Q_m(v), \nabla \varphi\right) + \left((\underbrace{v}_{= Q_m(v)} \cdot \nabla) Q_m(v), \varphi\right) = (f, \varphi)$$

(VII) Show that $v_m \xrightarrow[m\to\infty]{} v \in V_0$

$v_m \in V_m$

(VII) Show uniqueness of the velocity

subtract the equations for solutions $v_1$ and $v_2$

$\Downarrow$

$$\frac{1}{Re} \|\nabla(v_1-v_2)\|^2 \leq c^2 \|\nabla(v_1-v_2)\|^2 Re \|f\| \qquad / \cdot Re$$

$\Leftrightarrow$

$$\|\nabla(v_1-v_2)\|^2 - c^2 Re^2 \|f\| \cdot \|\nabla(v_1-v_2)\|^2 \leq 0$$

$$\Leftrightarrow \quad \|\nabla(v_1-v_2)\|^2 \underbrace{\left(1 - c^2 Re^2 \|f\|\right)} \leq 0$$

$$c^2 R e^2 \|f\| < 1 \iff \begin{cases} \text{if } \left(1 - c^2 R e^2 \|f\|\right) > 0 \\ \Rightarrow \|\gamma(v_1 - v_2)\| \le 0 \Rightarrow v_1 = v_2 \end{cases}$$

$\square$