Numerische Methoden in der Strömungsmechanik
OvGU WiSe 2023/24

Practical Exercises 4

In this problem sheet, we consider the linear stationary Stokes equations

$$\begin{aligned} \nabla \cdot v &= 0 \qquad \text{in } \Omega, \\ -\nu\Delta v + \nabla p &= f \qquad \text{in } \Omega, \\ v &= g \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

which is a good approximation of the Navier-Stokes equations for viscous fluids like honey. [1] The parameter $\nu \in \mathbb{R}^+$ describes the viscosity of the fluid.

Gascoigne uses bilinear or biquadratic finite elements for the calculations. Usually, we discretize the equations with the same elements for both pressure p and for the velocity vector v. Without stabilization this leads to instabilities since the discrete *“inf-sup”*-condition is not fulfilled.

Problem 4.1

In the weak formulation, the Stokes equations are given by

$$\begin{aligned} (\mathrm{d}iv\, v, \psi) &= 0 \qquad \forall \psi \in \mathcal{Q}, \\ \nu(\nabla v, \nabla \phi) - (p, \mathrm{d}iv \phi) &= (f, \phi) \quad \forall \phi \in \mathcal{V}. \end{aligned}$$

The second equation is to be understood componentwise, i.e.

$$\nu(\nabla v_1, \nabla \phi_1) + \nu(\nabla v_2, \nabla \phi_2) - (p, \partial_x \phi_1) - (p, \partial_y \phi_2) = 0.$$

Combining all equations we write $a(U, \Phi) = 0$, where $U = (p, v_1, v_2)$ and $\Phi = (\psi, \phi_1, \phi_2)$.

The bilinear form $a(U, \Phi)$ has to be implemented in the functions `Form` and `Matrix`. In `Form` the values `b[0]`, `b[1]` and `b[2]` contain all the contributions from the test functions ψ, ϕ_1 and ϕ_2 respectively.

```
void Form(VectorIterator b, const FemFunction &U,
                      const TestFunction &N) const
{
  b[0] += (U[1].x()+U[2].y())*N.m();

```

[1] The reason for the unusual order of the equations is that $\nabla \cdot v = 0$ is tested, in the weak formulation, with the test functions associated to the pressure. The pressure is the first solution component in Gascoigne, which makes the divergence condition the first equation internally.

```
  b[1] += _visc * (U[1].x() * N.x() + U[1].y() * N.y());

  b[2] += _visc * (U[2].x() * N.x() + U[2].y() * N.y());
}
```

(a) Specify the number of components in `stokes.h` and `local.h` in the function `GetNcomp()`. First, run the code without any stabilization. Have a look at the results with `paraview`. Can you explain your observations?

(b) As next, add a simple first order term to the bilinear form, e.g. (p, ψ) and check the results with `paraview` again.

(c) Now, add the simple stabilization term

$$S_h(p, \psi) = \alpha(h^2 \nabla p, \nabla \psi)$$

(*artificial diffusion*) to the divergence equation

$$(\mathrm{d}iv\ v, \psi) + S_h(p, \psi) = 0 \quad \forall \psi \in \mathcal{Q}.$$

To get the cellsize h use the `point` function and store the value in a `mutable` variable. Compare the results for different values of α.

Problem 4.2 Modify the right-hand side f and boundary data g such a way that the exact solution of the system (1) is given by

$$\begin{aligned} p(x,y) &= x^2 y^2 - \frac{1}{9}, \\ v_1(x,y) &= \sin(\pi x)\sin(\pi y), \\ v_2(x,y) &= \cos(\pi x)\cos(\pi y). \end{aligned}$$