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Exercise 2 for the lecture Fluid-structure Interactions
Summer 2025

Task 1

We consider the following three deformation fields (translation, rotation, sheering)

$$u_1(\mathbf{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad u_2(\mathbf{x}) = \begin{pmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad u_3(\mathbf{x}) = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}.$$

Compute, for each of the three deformation fields, the deformation gradient $F = I + \nabla u$, the Green Lagrange strain tensor $E = \frac{1}{2}(F^T F - I)$ and the linearized strain tensor $e = \frac{1}{2}(\nabla u + \nabla u)$.
Show that the rotation field yields zero strain when the nonlinear tensor is used but non-zero strain in the linear case.

Task 2

Let $A, B \in \mathbb{R}^{3\times 3}$ and $x, y \in \mathbb{R}^3$ and $u, v : \mathbb{R}^3 \to \mathbb{R}^3$.
Show, that

a) Show that, for the tensor contraction defined as

$$A : B = \sum_{i,j=1}^{3} A_{ij} B_{ij},$$

it holds ($tr(\cdot)$ is the trace)

$$A : B = tr(A^T B) = tr(AB^T)$$

b) Show that for the directional derivative

$$(v \cdot \nabla)v = \sum_{i=1}^{3} v_i \partial_i v$$

it holds

$$(v \cdot \nabla)v = \nabla v v.$$

Hint: By $\nabla v : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ *we denote the matrix field of the Jacobian. The product* $\nabla v v = \nabla v * v$ *is the matrix vector product.*

c) Let $F = I + \nabla u$, $J = \det(F)$ and assume that $u \in C^2$ such that the second derivatives commute, i.e.

$$\partial_i \partial_j u = \partial_j \partial_i u.$$

Show that

$$\mathrm{div}(JF^{-1}v) = JF^{-1} : \nabla v^T$$

Hint: The 3d case is difficult, see Lemma 2.61. Show the 2d case. Here you can use the simple rule for the inverse,

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

noting that $J = det(F)$. This result is important since it shows that $div(JF^{-1}v)$ can be evaluated based on first derivatives (of u) only although on first sight second derivatives are required.

The tasks will be discussed on May 6